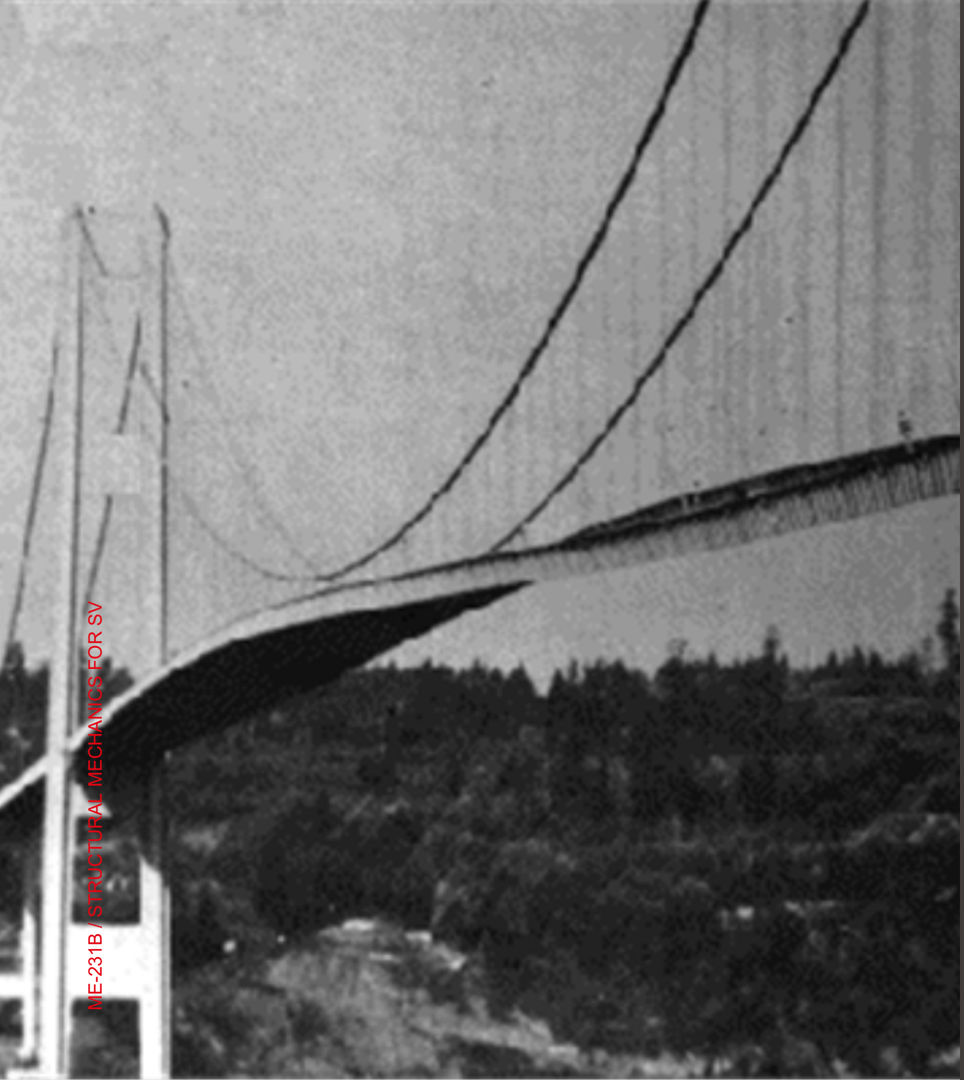


Week 7: Torsion

1. Statically indeterminate problems in torsion
2. Torsional stress concentration
3. Torsion of a solid non-circular member
4. Torsion of an inelastic member

Normal load vs. Torsion

	Spring	Bar in Tension	Bar in Torsion
Geometric property	Δx	δ	ϕ
Materials property	$N.A.$	E	G
Hoooke's Law	$F = k\Delta x$	$P = \frac{EA}{L}\delta$	$T = \frac{GJ}{L}\phi$
Strain distribution	$N.A.$	$\tau \neq \tau(r)$	$\tau(r) = \frac{Tr}{J}$

Static indeterminacy in Torsion

When a system is statically indeterminate we distinguish between two cases:

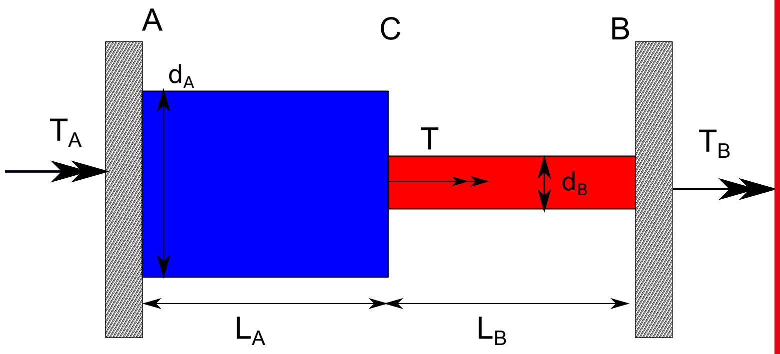
- *Internal indeterminacy*: here there is one more static member in the system than required to ensure static equilibrium.
- *External indeterminacy*: Here there is one more external reaction force than required to ensure static equilibrium.

Statically indeterminate torsion problems - Displacement stiffness method

$$k_T = \frac{JG}{L}$$

- We can also treat statically indeterminate systems in torsion with the the displacement stiffness method in matrix form.
- We can make use of the same approach we developed for the statically indeterminate bars, but now using the torsional stiffness:

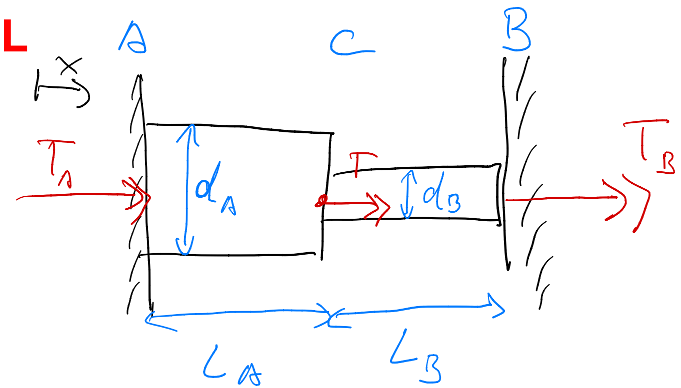
Example: External indeterminacy



A composite bar consists of two sections with diameter d_A and d_B respectively is clamped on both sides at walls (points A and B). A torque T acts on the point where the thickness changes (point C).

Calculate:

- An expression for the reaction torques at point A and B
- The maximum shear stresses in the bar sections AC and CB
- The angle of twist at point C



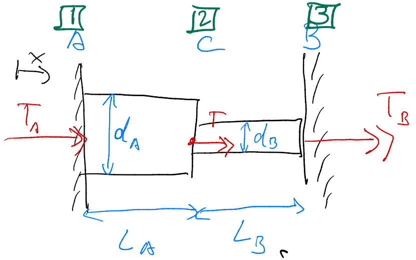
GIVEN:
- GEOMETRY: d_A, d_B, L_A, L_B
- LOAD: T
- MAT. PROP: $G_1 = G_2$

ASKED:
a) T_A, T_B
b) $\tau_{MAX}^{AC}, \tau_{MAX}^{CB}$
c) ϕ_C

GOV. PRINC.: D.S. METHOD IN TORSION

$$k_T = \frac{JG}{L} \qquad T_i = k_i \phi_i \qquad \tau_{MAX} = \frac{Tc}{J}$$

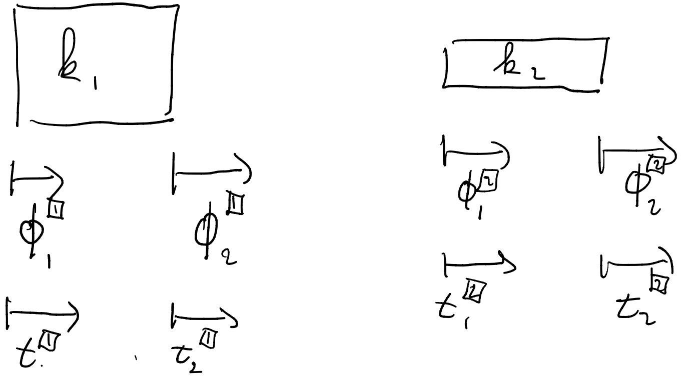
Answer:



External indeterminacy of degree 1

Number of nodes: 3

Number of sections: 2



Local stiffness matrix:

$$k_1: \begin{pmatrix} t_1^{[1]} \\ t_2^{[1]} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} \phi_1^{[1]} \\ \phi_2^{[1]} \end{pmatrix}$$

$$k_2: \begin{pmatrix} t_1^{[2]} \\ t_2^{[2]} \end{pmatrix} = \begin{pmatrix} k_2 & k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \phi_1^{[2]} \\ \phi_2^{[2]} \end{pmatrix}$$

Mapping local coordinates to global

Angles: $\phi_2^{[1]} = \phi_1^{[2]} = \Phi_C$

$\phi_1^{[1]} = \Phi_A$

$\phi_2^{[2]} = \Phi_B$

Torques: $t_1^{[1]} = T_A$

$T_C = t_1^{[2]} + t_2^{[1]}$

$T_B = t_2^{[2]}$

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ASSEMBLE THE GLOBAL STIFFNESS MATRIX

$$\begin{pmatrix} T_A \\ T_C \\ T_B \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \Phi_A \\ \Phi_C \\ \Phi_B \end{pmatrix} \qquad \text{EQN } \underline{1}$$

BOUNDARY CONDITIONS:

$$\Phi_A = 0 = \Phi_B \qquad T_C = T$$

$$\begin{pmatrix} T_A \\ T \\ T_B \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{pmatrix} 0 \\ \Phi_C \\ 0 \end{pmatrix} \qquad \text{EQN } \underline{2}$$

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$$T = (k_1 + k_2)\,\Phi_c$$

$$\Phi_c = \frac{1}{k_1 + k_2} T \qquad k_1 = \frac{J_A G}{L_A} \qquad k_2 = \frac{J_B G}{L_B}$$

$$\boxed{\Phi_c = \frac{1}{G} \cdot \frac{T}{\frac{J_A}{L_A} + \frac{J_B}{L_B}} = \frac{L_A L_B}{(J_A L_B + J_B L_A)G} \cdot T}$$

$$T_A = -k_1 \Phi_c = -\frac{J_A G}{L_A} \frac{L_A L_B}{(J_A L_B + J_B L_A)G} \cdot T = -\frac{J_A L_B}{J_A L_B + J_B L_A} T$$

$$T_B = -k_2 \Phi_c = -\frac{J_B L_A}{J_A L_B + J_B L_A} T$$

MAXIMUM SHEAR STRESSES IN AC & CB:

$$\tau_{MAX}^{AC} = \frac{T_A \cdot c}{J_A} = \frac{T_A \cdot d_A}{J_A \cdot 2} = \frac{L_B d_A}{2(J_A L_B + J_B L_A)} \cdot T$$

$$\tau_{MAX}^{CB} = \frac{T_B \cdot d_B}{J_B \cdot 2} = \frac{L_A d_B}{2(J_A L_B + J_B L_A)} \cdot T$$

Statically indeterminate torsion problems

- When we encounter *internal* static indeterminacy in a composite shaft of two or more tubes or materials together, we can use the displacement stiffness method (all shafts have same angle of twist)
- k_t: torsional stiffness

$$k_t = \frac{T}{\phi} = \frac{JG}{L} \qquad [k_t] = \frac{Nm}{rad}$$

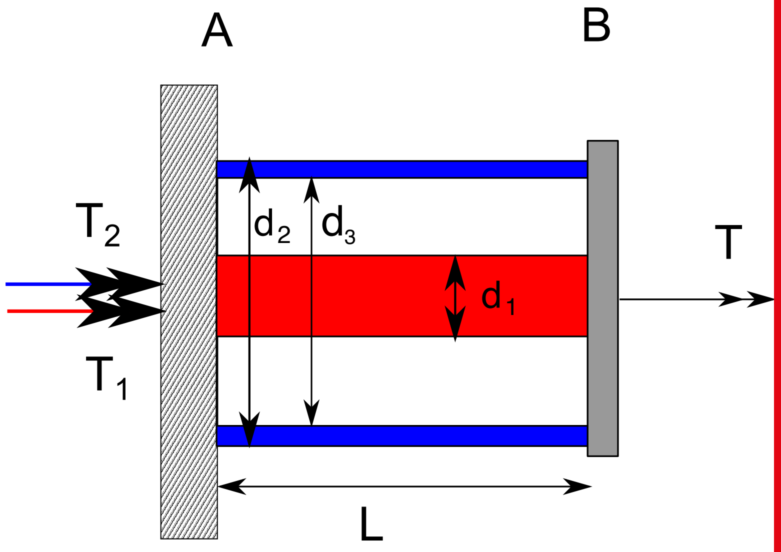
- The torque for the i^th shaft is then:

$$T_i = (k_t)_i \cdot \phi_i$$

- The total torque is then:

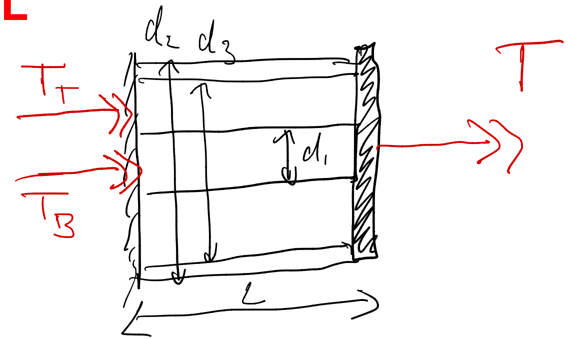
$$T = \sum_i (k_t)_i \cdot \phi_i$$

Example: Internal indeterminacy



A composite bar consists of a bar with an outer diameter d_1 and tube with an outer diameter d_2 joined at one end and both rigidly connected to the wall. Derive

- a formula for the angle of twist for the composite and
- the formula for the reaction torques at the wall acting on the tube and the bar.



GIVEN:

- GEOMETRY: ∘ BAR IN TUBE
  - ∘ $L_B = L_B = L$
  - ∘ d_1, d_2, d_3
- LOAD: T ACTS ON FREE END
  - T_B REACTION TORQUE ON BAR
  - T_T " " " TUBE
- MATERIAS: $G_T = G_B = G$

ASKED:

$\underline{a}$ Φ

$\underline{b}$ T_B, T_T

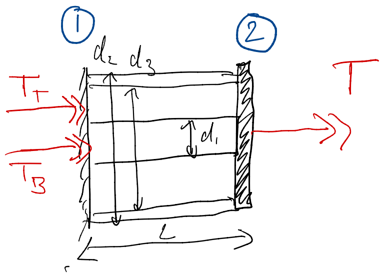
GOV. PRINC:

DS METHOD IN TORSION

$k = \frac{JG}{L}$ $T_i = k_i \phi_i$

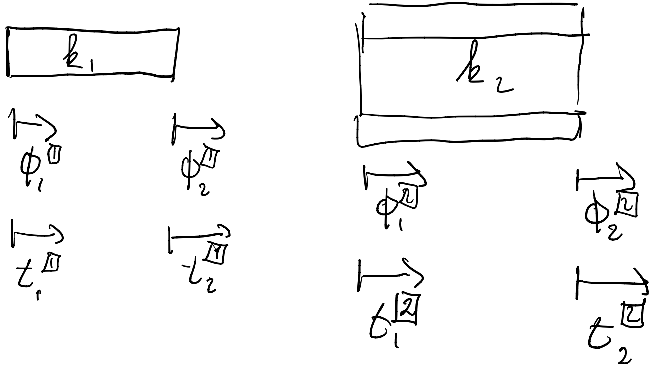
ANSWER:

- INTERNAL STATIC INDETERMINACY OF DEGREE: 1



\# NODES: 2

\# SEGMENTS: 2



LOCAL STIFFNESS MATRIX:

$$k^{[1]}: \begin{pmatrix} t_1^{[1]} \\ t_2^{[1]} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} \phi_1^{[1]} \\ \phi_2^{[1]} \end{pmatrix}$$

$$k^{[2]}: \begin{pmatrix} t_1^{[2]} \\ t_2^{[2]} \end{pmatrix} = \begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \phi_1^{[2]} \\ \phi_2^{[2]} \end{pmatrix}$$

MAPPING OF LOCAL COORDINATES TO GLOBAL.

- ANGLES
$$\phi_1^{[1]} = \phi_1^{[2]} = \Phi_A$$
$$\phi_2^{[1]} = \phi_2^{[2]} = \Phi_B$$

- TORQUES:
$$T_A = t_1^{[1]} + t_1^{[2]}$$
$$T_B = t_2^{[1]} + t_2^{[2]}$$

ASSEMBLY OF GLOBAL STIFFNESS MATRIX:

$$\begin{pmatrix} T_A \\ T_B \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & -(k_1 + k_2) \\ -(k_1 + k_2) & k_1 + k_2 \end{pmatrix} \begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix}$$

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B.C.:

$$T_B = T$$

$$\phi_A = 0$$

$$\begin{pmatrix} T_A \\ T \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & -k_1 + k_2 \\ -(k_1 + k_2) & k_1 + k_2 \end{pmatrix} \begin{pmatrix} 0 \\ \Phi_B \end{pmatrix}$$

$$T = (k_1 + k_2)\, \Phi_2 \quad \Rightarrow \quad \underline{\Phi_2 = \frac{1}{k_1 + k_2}\, T}$$

$$\sum M = 0 \quad \Rightarrow \quad T_A = -T = T_{BAR} + T_{TUBE}$$

$$T_{BAR} = t_1^{[1]} = -k_1 \phi_2^{[1]} = -k_1 \Phi_B = -\frac{k_1}{k_1 + k_2} T$$

$$T_{TUBE} = t_1^{[2]} = -k_2 \phi_2^{[2]} = -k_2 \Phi_B = -\frac{k_2}{k_1 + k_2} T$$

$$k_1 = \frac{J_1 G}{L} \qquad k_2 = \frac{J_2 G}{L} \qquad k_1 + k_2 = \frac{(J_1 + J_2) G}{L}$$

$$\Phi_B = \frac{L}{(J_1 + J_2) G} T \qquad T_{BAR} = -\frac{J_1}{J_1 + J_2} T \qquad T_{TUBE} = -\frac{J_2}{J_1 + J_2} T$$

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$$J_1 = \pi \frac{d_1^4}{32}$$

$$J_2 = \frac{\pi}{32}\left(d_2^4 - d_3^4\right)$$

$$\Phi_B = \frac{L}{(J_1 + J_2)G} T = \frac{LT}{G} \cdot \frac{1}{\frac{\pi}{32}\left(d_1^4 + d_2^4 - d_3^4\right)}$$

$$T_{BAR} = -\frac{J_1}{J_1 + J_2} T \qquad T_{TUBE} = -\frac{J_2}{J_1 + J_2} T$$

$$T_{BAR} = -\frac{\frac{\pi}{32} d_1^4}{\frac{\pi}{32}\left(d_1^4 + d_2^4 - d_3^4\right)} \cdot T$$

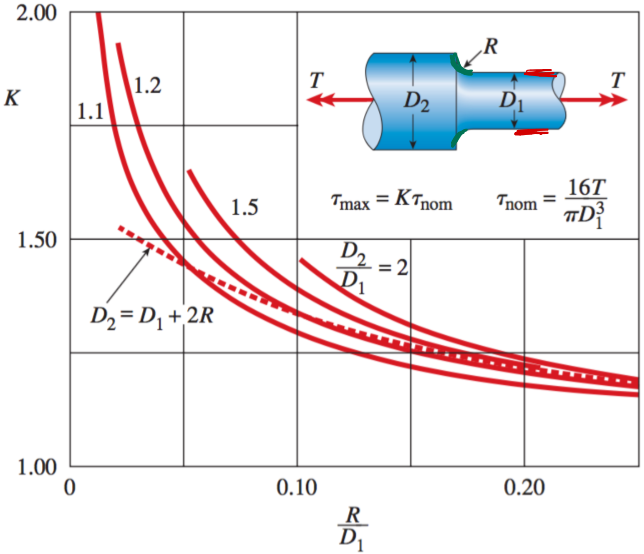
$$T_{TUBE} = \frac{d_2^4 - d_3^4}{d_1^4 + d_2^4 - d_3^4} T$$

EPFL

Stress concentration in torsion

In shafts with abrupt changes in dimension, large stress concentrations can occur

In a similar way to the stress concentration in tension, we can use a stress concentration factor to estimate the stress concentration



The maximum shear stress in torsion is then:

$$\tau_{max} = K \cdot \underbrace{\frac{T c}{J}}_{\text{smaller shaft}}$$

$\tau_{max} = \frac{Tc}{J}$

Transmission of power by a shaft

- Work: the energy developed by a force acting over a distance against a resistance
  - Linear distance: $W = F \cdot \Delta x$
  - Rotational distance: $W = T \cdot \Delta\theta \quad [\theta] = rad$

- Power: work done by unit of time:

$$P = \frac{T\theta}{t} = \omega T \quad [\omega] = \frac{rad}{s} \quad [P] = \frac{Nm}{s} = W\ (Watt)$$

- Often used unit: horsepower (hp)

$$1hp = 33.000\frac{ft \cdot lb}{min} = 550\frac{ft \cdot lb}{s} = 6600\frac{in \cdot lb}{s} = 745.7W$$

Common Imperial Units

1 in = 2.54cm

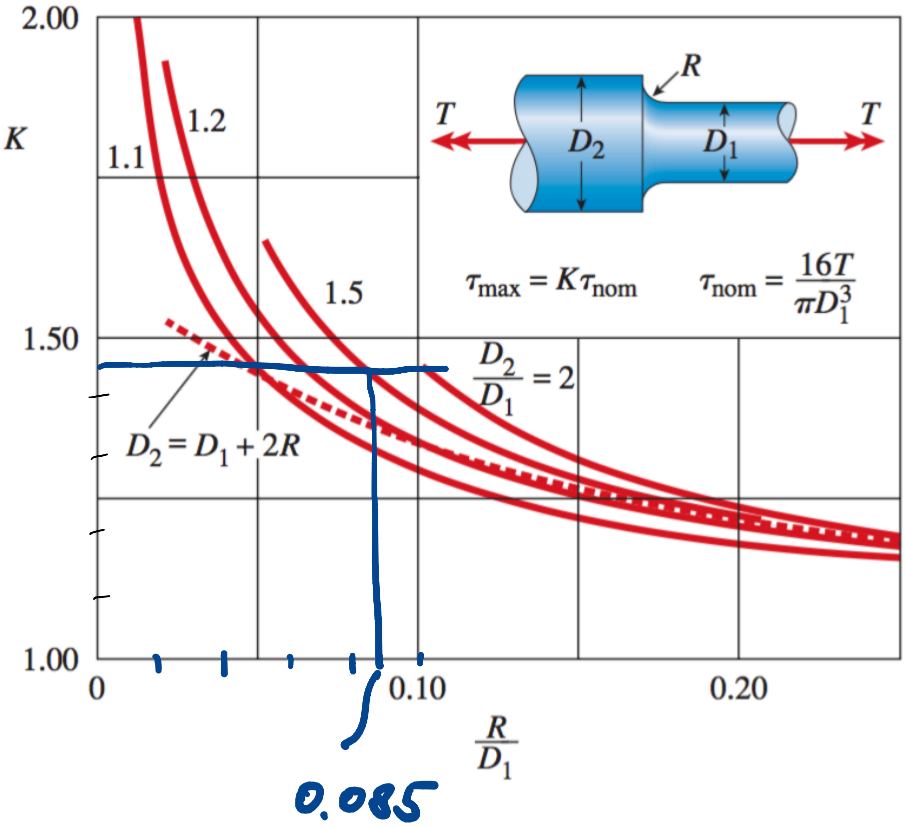
1 ft =12 in = 0.3048 m

1 lb = 16 oz = 0.4536kg

1 gallon = 3.7854 l

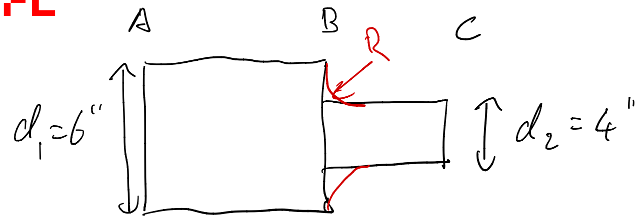
1 psi = 1 pound/in^2 = 6894.8 Pa

1hp = 300 (ft*lb)/min = 39600 (in*lb)/min= 745.7W



Examples for Torsion: Torsional stress concentration

Find the required fillet radius for the juncture of a 6-in.-diameter shaft with a 4-in.-diameter segment if the shaft transmits 110 hp at 100 rpm and the maximum shear stress is limited to 8000 psi.



given:
- GEOMETRY $d_1 = 6$ in
  $d_2 = 4$ in
- APPLIED POWER $P = 110$ Hp
- ROTATION SPEED: 100 rpm
- $\tau_{MAX} = 8000$ psi

ASKED: WHAT IS THE MINIMUM FILLET RADIUS r SO THAT THE SHAFT DOES NOT BREAK.

$$\tau_{MAX} = K \frac{Tc}{J_{SMALL\ SHAFT}} \qquad J = \frac{\pi}{2} c^4$$

GOV. PRINC: TORSIONAL STRESS CONCENTRATION:

$$P = \frac{T\theta}{t} = \omega T \Rightarrow T = \frac{P}{\omega}$$

ANSWER:

$$T = \frac{P}{\omega} = \frac{110\,HP \cdot 745.7\,W/HP}{100\,rpm \cdot \frac{1}{60}\frac{s}{min} \cdot 2\pi \frac{rad}{rot}} = 7833\,Nm$$

$$K = \frac{\tau_{MAX} \cdot J}{T \cdot c}$$

$1\,psi = 6894.75\,N/m^2 \approx 6.9\,kPa$

$c = 2\,in = 0.0508\,m$

$\tau_{MAX} = 55.2\,MPa$

$J = \frac{\pi}{2}(0,0508\,m)^4 = 1.04 \cdot 10^{-5}\,m^4$

$$K = \frac{55.2 \cdot 10^6 \cdot 1.04 \cdot 10^{-5}}{7833 \cdot 0.0508} = 1.44$$

$$\frac{R}{d_2} = 0,085 \qquad \underline{\underline{R_{min}}} = 4\,in \cdot 0,085 = \underline{\underline{0,34\,in}}$$

Torsion of a rectangular bar

- In our derivation for the torsion formula we have assumed that plane sections remain plane in torsion. This is only true for members with infinite axial symmetry (such as round bars or tubes).
- In a rectangular bar, there is no such symmetry and the cross sections will deform.
- For rectangular bars of length *L* and sides *a&b (a>b)* we can use the formulas:

$$\tau_{max} = \frac{T\,L}{C_1\,a\,b^2}$$

- And the torsion formula:

$$\phi = \frac{T\,L}{C_2\,a\,b^3 G}$$

- The torsional stiffness is then:

$$k_t = \frac{T}{\phi} = C_2 a b^3 \frac{G}{L}$$

Torsion of a rectangular bar

- For rectangular bars of length *L* and sides *a&b (a>b)* we can use the formulas:

$$\tau_{max} = \frac{TL}{C_1\, a\, b^2}$$

- And the torsion formula:

$$\phi = \frac{TL}{C_2\, a\, b^3 G}$$

- The torsional stiffness is then:

$$k_t = \frac{T}{\phi} = C_2 a b^3 \frac{G}{L}$$

a/b	C_1	C_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.321	0.312

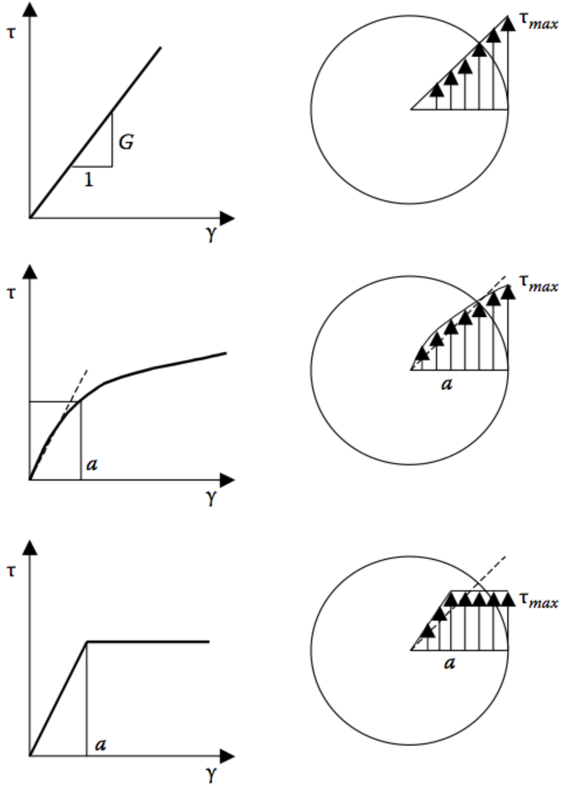
Torsion of an inelastic circular member

- If we are beyond the linear elastic regime, our torsion formula is no longer correct (we've assumed Hooke's law during its derivation)!
- However, the assumption that shear strain is linear with r still holds:

$$\gamma(r) = \frac{r}{c}\gamma_{max}$$

- Since we no longer have Hooke's law, we have to use the torsional stress-strain diagram of the material to find the shear stress
- For each r we can calculate the corresponding γ, and from the stress-strain curve we can read the stress τ at each r. We can plot then τ as a function of r

Torsion of an inelastic circular member



- With the known stress distribution we can then calculate the internal resisting torque at any given cross-section:

$$T = \int_A (\tau dA) r$$

- For a circular cross section:

$$T = 2\pi \int_0^c r^2 \tau(r) dr$$

- If there is no analytical expression for τ(r), we have to solve the integral numerically

Torsion of an inelastic bar: Ultimate torque

- *Ultimate torque T_U*: the maximum torque a member can take before failure
- We can calculate T_U by setting $\tau_{max} = \tau_U$ and do the integration or do a torsion experiment by twisting the circular member until it breaks.